Strategies for solving Trigonometrical Equations

Trigonometrical equations can be categorised into **7 main groups**, of which 2 have sub-variations.

The groups are:

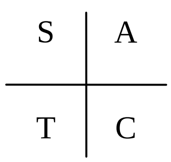
- Type 1: Simple re-arrangement
- Type 2: Extended domain
- Type 3: Change to Tangent
- Type 4: Quadratic
  - Simple Power
  - Common Factor
  - Trinomial
- Type 5: Double Angle
  - Reduces to common factor
  - Reduces to Trinomial
- Type 6: Phase Angle
- Type 7: Wave Function

In all cases the aim is to arrive at an expression of the form:

sin x = ….   cos x = ….   tan x = …..

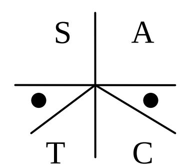
Which will result in **two** values for x.

These values are obtained by looking at the **sign** of the trig. function and using **All Sinners Take Care**.



e.g. sin x = -0.5   key $\sin^{-1}$ (0.5) and the calculator gives the acute angle of 30°

You then interpret this as follows:



sin x is **NEGATIVE**   so x will lie in the 3rd or 4th quadrants and will be the angle between the **rotating line** and the **x-axis**.

So the solution in this case is: x = 180° + 30° or 360° – 30° hence **x = 210°** or **x = 330°**

The methods and strategies used, apply equally to radians and degrees.

Recall: 360° = 2π radians or 180° = π radians

A table of Exact Values and compound angle formulae are shown below

Degrees	**30°**	**45°**	**60°**
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

sin (A + B) = sin A cos B + cos A sin B
sin (A – B) = sin A cos B – cos A sin B
cos (A + B) = cos A cos B – sin A sin B
cos (A – B) = cos A cos B + sin A sin B
sin 2A = 2 sin A cos A
cos 2A = $\cos^2 A - \sin^2 A$ = $2\cos^2 A - 1$ = $1 - 2\sin^2 A$
$\cos^2 A + \sin^2 A = 1$
tan A = $\frac{\sin A}{\cos A}$

We will now look at each of the **7 types** and the **strategies for solution.**

Type 1: Simple re-arrangement

e.g. $2 \sin x = 1$ $\quad\quad$ $\sqrt{3} \tan x + 1 = 0$ $\quad\quad$ $\cos x = 0.5$

These can be recognised by the fact that all it takes is a simple algebraic re-arrangement to obtain $\sin x = …..$ $\quad \cos x = …..$ $\quad \tan x = …..$

$2 \sin x = 1$	$\sqrt{3} \tan x + 1 = 0$	$\cos x = 0.5$
$\sin x = 0.5$	$\tan x = - 1/\sqrt{3}$	$\cos x = 0.5$ (already in required form)
acute $x = 30°$	acute $x = 30°$	acute $x = 60°$

Using ASTC gives

$x = 30°$ or $150°$	$x = 150°$ or $330°$	$x = 60°$ or $300°$
(1st & 2nd quadrants)	(2nd and 4th quadrant)	(1st & 4th quadrant)

Type 2: Extended domain

e.g. $\sin 2x = - 1/\sqrt{2}$ $\quad\quad$ $0 \leq x \leq 360°$

These can be recognised by the '2x', and no other trig. function involved.
These should not be confused with the double angle type, where another trig. term is present.

$\sin 2x = - 1/\sqrt{2}$

acute $2x = 45°$ $\quad\quad$ However $\quad 0 \leq 2x \leq 720°$ $\quad$ we need to consider twice the domain of x

Using ASTC gives
$2x = 225°$ or $315°$ or $585°$ or $675°$ $\quad$ (3rd and 4th quadrants in first and second revolution)

Thus $x = 112.5°$, $157.5°$, $292.5°$ or $337.5°$

Type 3: Change to tangent

e.g. $3\sin x = 2\cos x$ $\quad$ or $\quad$ $3\sin x – 2\cos x = 0$ $\quad\quad$ $0 \leq x \leq 360°$

These can be recognised by the presence of sin x and cos x only – with no constant term.

Divide both sides by cos x and use simple re-arrangement to give:

$\tan x = 2/3$

acute $x = 33.7°$

Using ASTC gives
$x = 33.7°$ or $213.7°$ (1st & 3rd quadrants)

Type 4: Quadratic

There are 3 forms of quadratic:

a) Simple power

e.g. $4\cos^2 x = 1$ or $4\cos^2 x - 1 = 0$ Recognise by squared term = constant

Re-arrange to give $\cos^2 x = \frac{1}{4}$
Take square roots of both sides – don't forget ± values

$\cos x = +1/\sqrt{2}$ or $\cos x = -1/\sqrt{2}$ note that there will be **up to 4 solutions**

acute x = 45°

Using ASTC gives
x = 45° or 315° (1st & 4th quadrants) and x = 135° or 225° (2nd & 3rd quadrants)

b) Common Factor

e.g. $\sin x \cos x - \sin x = 0$ Recognise by no constant term
and **sin x cos x** and either a **cos x** or **sin x** on its own.

Take out common factor

$\sin x(\cos x - 1) = 0$

So $\sin x = 0$

or $\cos x - 1 = 0$
thus $\cos x = 1$

Solve in the usual way – find acute angle then use ASTC
Note again there will be **up to 4 solutions**.

c) Trinomial

e.g. $4\sin^2 x + 11\sin x + 6 = 0$ Looks like a quadratic equation in either sin x, cos x or tan x

This type can appear in combinations of:
$\sin^2 x$ and $\sin x$, $\cos^2 x$ and $\sin x$, $\sin^2 x$ and $\cos x$ or $\cos^2 x$ and $\cos x$

If both cos and sin appear,
then use the identity $\mathbf{\sin^2 x + \cos^2 x = 1}$ to make the squared term the same as the other term.

From this point – put the equation into 2 brackets: $(4\sin x + 3)(\sin x + 2) = 0$

then equate each bracket to 0

so $4\sin x = 3$ $\sin x = \frac{3}{4}$
or $\sin x = -2$ not possible so discard this option.

Continue as previously, finding acute angle, then using ASTC.
Note again, that there will be **up to 4 solutions**

Type 5: Double Angle

There are 2 forms:

a) Reduces to common factor form

e.g. sin 2x – cos x = 0 Always contain **sin 2x** and **never** have a **constant term**.

Replace sin 2x using: sin 2x = 2 sin x cos x
and we obtain: 2 sin x cos x – cos x = 0

continue as you would with the common factor form of type 4.

b) Reduces to trinomial form

e.g. cos 2x + sin x = 0 Always contain cos 2x and either sin x or cos x
Constant term may or may not be present.

Replace cos 2x using: cos 2x = 2 $\cos^2$ x – 1 or 1 – 2 $\sin^2$ x

obtain: 1 – 2 $\sin^2$ x + sin x = 0
re-arrange to: 2 $\sin^2$ x – sin x – 1 = 0
Factorise into 2 brackets: (2 sin x + 1)(sin x – 1) = 0

Proceed as trinomial form of Quadratic type.

Type 6: Phase Angle

e.g. cos (x + 20) = 0.4 or cos(x + 20) – 0.4 = 0

Basically same as simple re-arrangement type 1

x + 20 = $\cos^{-1}$ 0.4

so acute (x + 20) = 66.4°

Using ASTC gives x + 20 = 66.4° or x + 20 = 293.6°

So x = 46.4° or 273.6°

Type 7: Wave Function

e.g. 2cos x + 3sin x = 1 Recognise by both **sin x** and **cos x** being present **AND** a **constant** term.

Put in the form Rcos (x ± α) or Rsin (x ± α) - you will be told which to do in the question.

Expand out using compound angles: (assume) R(cos x - α) = R cos x cos α + R sin x sin α

Compare coefficients, R cos α = 2 R sin α = 3

Square and add to get R: $R^2 = 2^2 + 3^2$ $R^2 = 13$ R = √13

Divide the sin by cos expression to get: tan α = 3/2 acute α = 56.3°

Use information from **R cos α = 2** and **R sin α = 3** to deduce **α is in 1st quadrant**

Now replace 2cos x + 3sin x with Rcos (x - α) using R = √13 and α = 56.3°

So we get: √13 cos(x - 56.3°) = 1 Re-arrange to get cos(x – 56.3°) = 1/√13

and proceed as Type 6: Phase Angle

Summary of types and examples

Type 1: Simple Re-arrangement

a) Sin x, cos x, tan x = 1, 0, -1, exact values, decimals, any constant
b) 2 tan x + 1 = 0 etc.

Type 2: Extended Domain

a) sin 2x, cos 3x = constant
b) sin 3x + 1 = 0 etc.

Type 3: Change to Tangent

a) sin x = cos x
b) 2 sin x = 3 cos x
c) 3 sin x – 2 cos x = 0

Type 4: Quadratic

(i) Simple power

a) $2\sin^2 x = 1$
b) $2\cos^2 x + 1 = 0$

(ii) Common Factor

a) sin x = sin x cos x
b) 3cos x – 2 sin x cos x = 0

(iii) Trinomial Form

a) $4 \sin^2 x + 11 \sin x + 6 = 0$
b) $5 \sin^2 x - 2 = 2 \cos x$
c) $6 \cos^2 x - \sin x - 5 = 0$

Type 5: Double Angle

(i) Reduces to Common Factor

a) Sin 2x – cos x = 0
b) Sin 2x + sin x = 0

(ii) Reduces to Trinomial form

a) Cos 2x + cos x = 0
b) cos 2x + sin x = 0
c) cos 2x – 7 sin x – 4 = 0
d) cos 2x – 5 sin x = 2

Type 6: Phase Angle

a) cos (x + 20) = 0.4 sin (x – 15) = $1/\sqrt{2}$
b) 2 sin (2x – 60) = 1
c) 3 cos (3x + 40) + 1 = 0

Type 7: Wave Function

a) cos x + sin x = 1
b) 2 cos x + 3 sin x = -1
c) 3 cos 2x + 4 sin 2x + 1 = 0

Mixed Examples: - Identify the type first

1. $2 \sin x + \sqrt{3} = 0$ $\quad 0 \le x \le 360°$

2. $6 \cos^2 x - \sin x – 5 = 0$ $\quad 0 \le x \le 360°$

3. $\cos 2x + 5 \cos x – 2 = 0$ $\quad 0 \le x \le 360°$

4. $\sin (x - \pi/2) = 0.5$ $\quad 0 \le x \le \pi$

5. $\sqrt{3} \tan\theta + 1 = 0$ $\quad 0 \le \theta \le 2\pi$

6. $\sin 2x – \cos x = 0$ $\quad 0 \le x \le 360°$

7. $\sin 3x = \sqrt{3}/2$ $\quad 0 \le x \le 360°$

8. $\sin 2\theta - \sin \theta = 0$ $\quad 0 \le \theta \le 2\pi$

9. $2 \sin (2x + \pi/3) = 1$ $\quad 0 \le x \le \pi$

10. $2 \cos 2x - 3\cos x + 1 = 0$ $\quad 0 \le x \le 360°$

11. $\tan^2 \theta + \tan \theta - 12 = 0$ $\quad 0 \le \theta \le 2\pi$

12. $\cos 3x = 0$ $\quad 0 \le x \le 360°$

13. $\sqrt{3} \cos x – \sin x = 0$ $\quad 0 \le x \le 360°$

14. $6 \cos 2\theta - 5\cos \theta + 4 = 0$ $\quad 0 \le \theta \le 2\pi$

15. $4 \cos^2 x – 1 = 0$ $\quad 0 \le x \le 360°$

16. $\cos x + \sin x = 1$ $\quad 0 \le x \le 360°$

17. $15\cos^2 \theta + 7 \cos \theta - 2 = 0$ $\quad 0 \le \theta \le 2\pi$

18. $4\cos (2x + 40) = 3$ $\quad 0 \le x \le 360°$

19. $3 \cos 2x + 4 \sin 2x + 1 = 0$ $\quad 0 \le x \le 180°$

20. $\cos \theta - \sqrt{3}/2 = 0$ $\quad 0 \le \theta \le 2\pi$

Answers	
1.	240°, 300°
2.	19°, 161°, 210°, 330°
3.	60°, 300°
4.	2.1 radians
5.	5π/6, 11π/6 radians
6.	30°, 90°, 150°, 270°
7.	20°, 40°, 140°, 160°, 260°, 280°
8.	0, π/3, π, 5π/3, 2π radians
9.	0.8, 2.9 radians
10.	0°, 104.5°, 255.5°, 360°
11.	1.82, 4.96, 1.25, 4.39 radians
12.	30°, 90°, 150°, 210°, 270°, 330°
13.	60°, 240°
14.	0.8, 1.8, 4.5, 5.4 radians
15.	60°, 120°, 240°, 300°
16.	0°, 90°, 360°
17.	1.37, 4.91, 2.30, 3.98
18.	1°, 139°, 181°, 319°
19.	77°, 156°
20.	π/6, 11π/6 radians